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APPLYING TAYLOR SERIES IN ONE VARIABLE TO FUNCTION LIMITS

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Abstract

It is common belief that the first person in history who ever used infinite series approximations to trigonometric functions was the Indian mathematician Madhava (1340–1425 bC), the founder of the Kerala astronomy school, who deduced some approximation formulas for trigonometric functions by geometric arguments.
In the West, the intuitive idea of more general Taylor series was given by the Scottish mathematician James Gregory, but they were formally introduced by the English mathematician Brook Taylor in 1715. Taylor series centered at 0 are also called Maclaurin series, in the honour of the Scottish mathematician Colin Maclaurin, who extensively used this special case in his works in the 18th century. The partial sums of a Taylor series are called Taylor polynomials. The Taylor series are very powerful methods used to function approximations (numerical approximations, integrals, differential equations, asymptotic calculus). In this article we use the asympthotic expressions of Taylor series in order to calculate function limits.

***Key words**: function approximation, function limits, Maclaurin series, Taylor polynomial, Taylor series.*

INTRODUCTION

First, we present without proof some classical derivability theorems.

Theorem 1 (*Fermat's theorem*, Boboc, 1998). Let $f:(a,b)\to\mathbb{R}$ be a derivable function. If $x_0\in(a,b)$ is an extremely local point of f, then $f'(x_0)=0$.

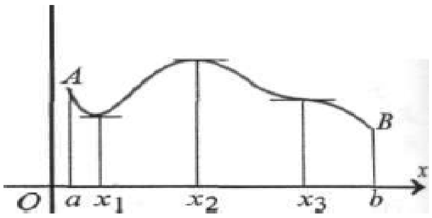

Figure 1. The geometrical interpretation of Fermat's theorem

The point x_1 is a minimum and the x_2 is a maximum on (a,b). The graph tangents at the abscissa points x_1 and x_2 are horizontal.

Theorem 2 (*Rolle's theorem*, Colojoară, 1983). Let $f:[a,b]\to R$ a continuous function, derivable on (a, b). If $f(a)=f(b)$, then there exists $c\in(a,b)$ such that $f'(c)=0$.

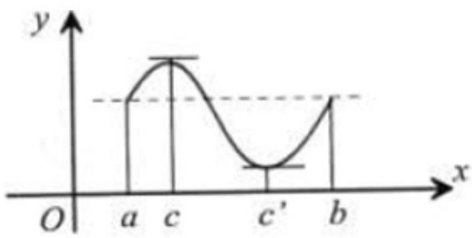

Figure 2. The geometrical interpretation of Rolle's theorem

Remark 1. This point c is not necessarily unique, as shown in the picture above.

Theorem 3 (*Lagrange's mean value theorem*). Let $f:[a,b]\to\mathbb{R}$ a continuous function, derivable on (a, b). Then there exists $c\in(a,b)$ such that $\frac{f(a)-f(b)}{a-b}=f'(c)$.
If we denote $A\big(a,f(a)\big)$ and $B\big(b,f(b)\big)$, then $\frac{f(a)-f(b)}{a-b}$ represents the slope of the line AB and $f'(c)$ represents the slope of the tangent to the graph in point $C\big(c,f(a)\big)$. Then, the conclusion of Lagrange's theorem is expressed geometrically by the existence of a point (at

least) where the tangent to the graph is parallel to the line joining the ends of the graph.

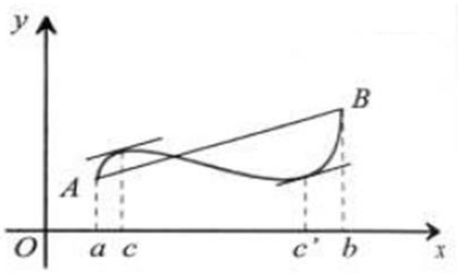

Figure 3. The geometrical interpretation of Lagrange's theorem

Theorem 4 (*Cauchy's mean value theorem*). Let $f: [a, b] \to \mathbb{R}$ a continuous function, derivable on (a, b), with $g'(x) \neq 0$, for all $x \in (a, b)$. Then there is $c \in (a, b)$ such that

$$\frac{f(a)-f(b)}{g(a)-g(b)} = \frac{f\prime(c)}{g\prime(c)}.$$

TAYLOR SERIES

Definition. Let $I \subset \mathbb{R}$ be an interval, $f \in C^n(I)$ (n times derivable on I, with the n-th derivative continuous) and $x_0 \in I$, fixed. It is called *the Taylor polynomial of degree n, attached to the function f at the point* x_0, the polynomial:

$$T_{n,f,\,x_0} = T_{n,f,\,x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$
$$= f(\,x_0) + f'(\,x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Remark 2. For simplicity, we will denote $T_{n,f,\,x_0}(x)$ by $T_n(x)$ or T_n.

Definition 1. With the previous notations, $R_n(x) = f(x) - T_n(x)$ is called *the reminder of order n* associated to the function f at the point x_0.

Remark 3. We have $f(x) = T_n(x) + R_n(x)$ and, if $\lim_{n\to\infty} R_n(x) = 0$, then

$$f(x) = \lim_{n\to\infty} T_n(x) = \sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$
$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

(the development in *Taylor series* centered at $x = x_0$).

Theorem 5 (Taylor's theorem with Lagrange's remainder, Nicolescu et al., 1971). Let $I \subset \mathbb{R}$ be an interval, $f \in C^{n+1}(I)$ and $x_0 \in I$. Then, there exists $c_x \in (x, x_0)$ such that:

$$f(x) = T_n(x) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$$

Remark 4. In fact $c_x \in (x, x_0)$ or $c_x \in (x_0, x)$, in a convenient order.

Proof. By mathematical induction. For $n = 0$, the statement becomes: $f(x) = f(x_0) + f'(c_x)(x-x_0)$, for some $c_x \in (x, x_0)$, which is true from Lagrange's theorem.

$n-1 \to n$: Suppose the statement is true for $n-1 \in \mathbb{N}$. Thus, for any function $g \in C^n(I)$, there exists $d_x \in (x, x_0)$ such that:

$$g(x) = T_{n-1}(x) + \frac{g^{(n)}(d_x)}{n!}(x-x_0)^n$$

We denote $U(x) = f(x) - T_n(x)$ and $V(x) = (x-x_0)^{n+1}$. We obtain:

$$\frac{f(x)-T_n(x)}{(x-x_0)^{n+1}} = \frac{U(x)}{V(x)} = \frac{U(x)-U(x_0)}{V(x)-V(x_0)} = \frac{U'(d_x)}{V'(d_x)}$$
$$= \frac{f'(d_x)-T_{n-1}(d_x)}{(n+1)(d_x-x_0)^n} = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1}$$

where $d_x \in (x, x_0)$ and $c_x \in (d_x, x_0) \subset (x, x_0)$ (we have applied the theorem of Cauchy for the functions $U(x)$ and $V(x)$ the induction hypothesis for $g = f' \in C^n(I)$).

Remark 5. Taylor's theorem is the generalization to the order n of Lagrange's theorem.

Definition 2. A Taylor series development at $x_0 = 0$ is called *Maclaurin series*:

$$f(x) = \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!}x^n$$
$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

The list of Maclaurin series for some common functions:

The Exponential function:

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots, \quad x \in \mathbb{R}$$

The Logarithmic function:

$$\ln(1+x) = \sum_{n\geq 1} (-1)^{n+1} \frac{x^n}{n}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad x \in (-1,1]$$

The Binomial function:

$$(1+x)^\alpha = \sum_{n\geq 0} C_n^\alpha x^n$$

$$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \ldots \quad, x \in (-1,1).$$

where $C_n^\alpha = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$.

Particularly, for $\alpha = \frac{1}{2}$ we obtain:

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}}$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^2 - \ldots \quad, x \in (-1,1).$$

The Trigonometric functions:

$$\sin x = \sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad, x \in \mathbb{R}.$$

$$\cos x = \sum_{n\geq 0} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad, x \in \mathbb{R}.$$

$$tg\, x = \sum_{n\geq 1} \frac{B_{2n} \cdot (-4)^n (1-4^n) x^{2n-1}}{(2n)!}$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \qquad, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

where B_n, $n \in N$, are the Bernoulli numbers, defined by the equality

$$\frac{x}{e^x - 1} = \sum_{n\geq 0} \frac{B_n}{n!} x^n$$

$$B_{2n+1} = 0, \qquad n \geq 1,$$

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42} \ldots$$

$$arctg\, x = \sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \qquad, x \in [-1,1].$$

The Hyperbolic functions:

$$sh\, x = \frac{e^x - e^{-x}}{2} = \sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!}$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \quad, x \in \mathbb{R}.$$

$$ch\, x = \frac{e^x + e^{-x}}{2} = \sum_{n\geq 0} \frac{x^{2n}}{(2n)!}$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \quad, x \in \mathbb{R}.$$

Remark 6. $ch^2 x - sh^2 x$=1.

ELEMENTS OF ASYMPTOTIC CALCULUS

Definition 3. Let $f, g: I \to \mathbb{R}$. We say that $g(x) = o(f(x))$ at $x = x_0$ if there exists $h: I \to \mathbb{R}$, such that $g(x) = f(x)h(x)$ and $\lim_{x\to x_0} h(x) = 0$.

Remark 7. If $f(x) \neq 0$ on a neighborhood of x_0 (excluding x_0), then

$$g(x) = o(f(x)) \Leftrightarrow \lim_{x\to x_0} h(x) = 0.$$

Properties:

1) $o(f(x)) = o(1) \quad \Leftrightarrow \lim_{x\to x_0} f(x) = 0$;

2) $o\big(f(x)\big) + o\big(f(x)\big) = o\big(f(x)\big)$;

3) $g(x)\cdot o\big(f(x)\big) = o(f(x)g(x))$;

4) $o(f(x))\cdot o\big(g(x)\big) = o\big((f(x)g(x)\big)$;

5) $\alpha\cdot o\big(f(x)\big) = o\big(f(x)\big),\ \forall\alpha\in\mathbb{R}^*$.

APPLICATIONS TO THE CALCULATION OF LIMITS

1) Prove that $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$.

Solution. We use the MacLaurin developement for the function $\sin x$ at $x_0 = 0$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = x + o(x)$$

Thus, $\frac{\sin x}{x} = \frac{x + o(x)}{x} = 1 + o(1) \underset{x\to 0}{\longrightarrow} 1$.

2) Compute $\lim\limits_{x\to 0}\frac{tg(x)-x}{x^3}$.

Solution. We use the Maclaurin developement for $\cos x$ at 0:

$$tg\ x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$$
$$= x + \frac{x^3}{3} + o(x^3)$$

So, we have

$$\frac{tg(x)-x}{x^3} = \frac{\frac{x^3}{3}+o(x^3)}{x^3} = \frac{1}{3} + o(1) \underset{x\to 0}{\longrightarrow}\frac{1}{3}.$$

3) Compute $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$.

Solution. We use the Maclaurin developement for $\cos x$ at 0:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$
$$= 1 - \frac{x^2}{2} + o(x^2)$$

Thus, we obtain

$$\frac{1-\cos x}{x^2} = \frac{\frac{x^2}{2}-o(x^2)}{x^2} = \frac{1}{2} - o(1) \underset{x\to 0}{\longrightarrow}\frac{1}{2}.$$

4) Compute $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}$.

Solution. We consider the Maclaurin developement for $\ln(1 + x)$ at the point 0:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$
$$= x + o(x)$$

Therefore, we have

$$\frac{\ln(1+x)}{x} = \frac{x+o(x)}{x} = 1 + o(1) \underset{x\to 0}{\longrightarrow} 1.$$

5) Compute $\lim\limits_{x\to 0}\frac{\sin x - arctg\ x}{x^3}$.

Solution. We use the Maclaurin developement for the *sin x* and *arctg x* at *0*:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$
$$= x - \frac{x^3}{6} + o(x^3)$$
$$arctg\ x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$
$$= x - \frac{x^3}{3} + o(x^3)$$

Therefore,

$$\frac{\sin x - arctg\ x}{x^3} = \frac{\frac{x^3}{3} - \frac{x^3}{6} + o(x^3) - o(x^3)}{x^3} =$$
$$\frac{\frac{x^3}{6}+o(x^3)}{x^3} = \frac{1}{6} + o(1) \underset{x\to 0}{\longrightarrow}\frac{1}{6}.$$

6) Show that $\lim\limits_{x\to 0}\frac{(1+x)^\alpha - 1}{x} = \alpha$, where $\alpha\in\mathbb{R}$.

Solution. We use the Maclaurin developement for the power series $(1+x)^\alpha$ at *0*:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \ldots$$
$$= 1 + \alpha x + o(x)$$

So, for x →0, we have

$$\frac{(1+x)^\alpha - 1}{x} = \frac{\alpha x + o(x)}{x} =$$
$$= \alpha + o(1) \underset{x\to 0}{\longrightarrow}\alpha.$$

7) Compute $\lim\limits_{x\to 0}\left[\frac{1}{x^2} - \frac{e^x}{(e^x-1)^2}\right]$.

Solution. We have that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots,$$

so

$$(e^x - 1)^2 =$$
$$= \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)$$
$$= x^2 + x^3 + \frac{7}{12}x^4 + o(x^4)$$

Thus, we obtain:

$$\frac{1}{x^2}-\frac{e^x}{(e^x-1)^2}=\frac{(e^x-1)^2-x^2e^x}{x^2(e^x-1)^2}=$$

$$=\frac{\frac{1}{12}x^4+o(x^4)}{x^4+o(x^4)}=\frac{\frac{1}{12}+o(1)}{1+o(1)}\xrightarrow[x\to 0]{}\frac{1}{12}$$

8) Calculate $\lim\limits_{x\to 0}\frac{sh(x^4)-x^4}{(x-\sin x)^4}$.

Solution. We know that

$$sh(x)=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}\ldots,$$

which implies that

$$sh(x^4)-x^4=\frac{x^{12}}{3!}+\frac{x^{20}}{5!}+\frac{x^{28}}{7!}+\ldots$$
$$=\frac{x^{12}}{6}+o(x^{12})$$

For the denominator, one has:

$$(x-\sin x)^4=\left[\frac{x^3}{6}+o(x^3)\right]^4$$
$$=x^{12}\left[\frac{1}{6}+o(1)\right]^4=\frac{x^{12}}{6^4}+o(x^{12})$$

Therefore, we obtain:

$$\frac{sh(x^4)-x^4}{(x-\sin x)^4}=\frac{\frac{x^{12}}{6}+o(x^{12})}{\frac{x^{12}}{6^4}+o(x^{12})}=$$
$$\frac{\frac{1}{6}+o(1)}{\frac{1}{6^4}+o(1)}\xrightarrow[x\to 0]{}6^3=216.$$

9) Compute $\lim\limits_{x\to 0}\frac{\sqrt{1+4x}-1-\sin(2x)}{\ln(1+x^2)}$.

Solution. We have the following Taylor developments at 0:

$$\sqrt{1+4x}=(1+4x)^{\frac{1}{2}}$$
$$=1+2x-2x^2+4x^3+\ldots$$
$$=1+2x-2x^2+o(x^2)$$

$$\sin(2x)=2x-\frac{4}{3}x^3+\ldots$$
$$=2x+o(x^2)$$

$$\ln(1+x^2)=x^2-\frac{x^4}{2}+\frac{x^6}{3}\ldots$$
$$=x^2+o(x^2)$$

We obtain:

$$\frac{\sqrt{1+4x}-1-\sin(2x)}{\ln(1+x^2)}=\frac{-2x^2+o(x^2)}{x^2+o(x^2)}$$

$$=\frac{-2+o(1)}{1+o(1)}\xrightarrow[x\to 0]{}-2.$$

10) Find the parameters $a,b\in\mathbb{R}$ for which $\lim\limits_{x\to\infty}\left(a\sqrt{x^2+ax}-b\sqrt{x^2+bx}-2x\right)=1$.

Solution. We replace $x\to\infty$ with $\frac{1}{y}$, where $y\to 0, y>0$. The initial limit transposes into the following limit where $x\to 0, x>0$:

$$\frac{1}{y}\left(a\sqrt{1+ay}+b\sqrt{1+by}-2\right)\xrightarrow[\substack{y\to 0\\ y>0}]{}1.$$

For the sake of simplicity, we denote again y by x. Thereby:

$$\frac{1}{x}\left(a\sqrt{1+ax}+b\sqrt{1+bx}-2\right)\xrightarrow[\substack{x\to 0\\ x>0}]{}1.$$

Using the Taylor developments at 0:

$$\sqrt{1+ax}=(1+ax)^{\frac{1}{2}}$$
$$=1+\frac{a}{2}x-\frac{a^2}{8}x^2+\ldots$$
$$=1+\frac{a}{2}x-\frac{a^2}{8}x^2+o(x^2)$$

$$\sqrt{1+bx}=(1+bx)^{\frac{1}{2}}$$
$$=1+\frac{b}{2}x-\frac{b^2}{8}x^2+\ldots$$
$$=1+\frac{b}{2}x-\frac{b^2}{8}x^2+o(x^2)$$

we must obtain:

$$\frac{1}{x}\left(a\sqrt{1+ax}+b\sqrt{1+bx}-2\right)$$
$$=\frac{a+b-2}{x}+\frac{a^2+b^2}{2}+o(x)\xrightarrow[\substack{x\to 0\\ x>0}]{}1$$

This is possible if and only if $\begin{cases}a+b-2=0\\ a^2+b^2=2\end{cases}$, which represents a system of equations with the solutions $a=b=1$.

CONCLUSION

Taylor series are a very powerful method to solve difficult function limits.

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REFERENCES

Boboc N., 1998, Analiză matematică (vol. 1), Ed.Universității din București

Colojoară I., 1983, Analiză matematică (vol. 1), Ed. Didactică și Pedagogică

Nicolescu M., Dinculeanu N., Marcus S., 1971, Analiză matematică (vol. 1), Ed. Didactică și Pedagogică

https://en.wikipedia.org/wiki/Taylor_series

https://mathworld.wolfram.com/BernoulliNumber.html