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# METHODS OF APPROXIMATING THE RIEMANN INTEGRALS AND APPLICATIONS

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***Abstract***

*Often, in practice, one reaches to incalculable integrals, but which can be approximated by numerical methods. In fact, in terms of application, there is no need for the exact result, but for knowing its value with an accuracy no matter how good. In this paper we present two methods to approximate Riemann integrals: the method of rectangles and trapezoids method. After reviewing the theoretical results, we consider some applications, focusing on the precision of approximations.*

***Key words***: *Riemann integral, rectangle method, trapezoidal method, approximation, precision*

## INTRODUCTION

Sometimes, in our practice work, we obtain integrals with incomputable primitives, called transcendental integrals. In some cases, these integrals can be calculated with more advanced techniques, such as complex analysis, but most of the times they can only be approximated by numerical methods. The most elementary methods are linked to Riemann sums: rectangles method, trapezoids method, Simpson's method, Hermite's method, Newton's method etc. In this article, we consider only first two methods. We begin by proving the formulas and then we apply them to several integrals, some of which have important applications.

## MATERIALS AND METHODS

We follow (Grigore, 1990), (Munteanu and Stanica, 2006) and (Rosca, 2000) for the proofs.

The rectangles method

In mathematics, especially in integral calculus, the rectangle method (also called the midpoint or mid-ordinate rule) computes an approximation to a definite integral, made by finding the area of a collection of rectangles whose heights are determined by the values of the function. Specifically, the interval $[a,b]$ on which the function has to be integrated is divided into n equal subintervals of length $h=\frac{b-a}{n}$.

The rectangles are then drawn so that either their left or right corners, or the middle of their top line lies on the graph of the function, with bases running along the O$x$-axis.

This process is illustrated by the next figures: Figure 1, Figure 2, Figure 3. (see http://www.maa.org/external_archive/joma/Volume7/Aktumen/Rectangle.html and http://www.mathcs.emory.edu/~cheung/Courses/170/Syllabus/07/rectangle-method.html).

Inner Rectangles. For the lower sum, corresponding to inner rectangles, we use Leftbox approximation on the interval [−1, 0] and the Rightbox approximation on the interval [0, 1].

Middle Rectangles. To obtain middle rectangles, we simply use Middlebox approximation on the entire interval [−1, 1].

Outer Rectangles. For the upper sum, corresponding to outer rectangles, we use Rightbox approximation on the interval [−1,0] and the Leftbox approximation on the interval [0, 1].

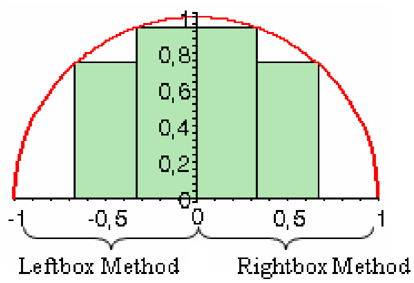



Figure 1. The inner rectangle approximation

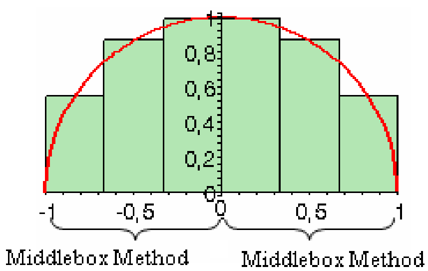



Figure 2. The middle rectangle approximation

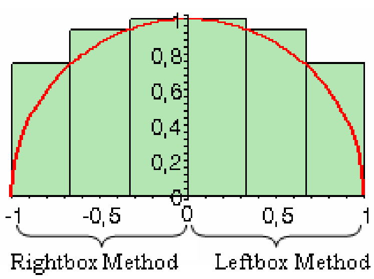



Figure 3. The outer rectangle approximation

The approximation to the integral is then calculated by adding up the areas (base multiplied by height) of the n rectangles, giving the formula:

$$\int_a^b f(x)dx = h\cdot\sum_{i=0}^{n-1} f(x_i)$$

where $h=\frac{b-a}{n}$ and $x_i = a+ih$.

The formula for xi above gives xi for the Top-left corner approximation.

A graphic representation of the rectangle method we can follow in the next figures:

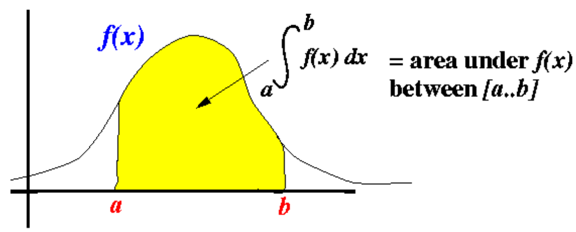



Figure 4. The domain of studing the area of one function

The next step for us is to divide the specified area (the coloured one) in subintervals (rectangles) . The more rectangles we have, the better the approximation is. For this aspect, the rectangles are then drawn so that either their left or right corners, or the middle, of their top line lies on the graph of the function, with bases running along the O$x$-axis:

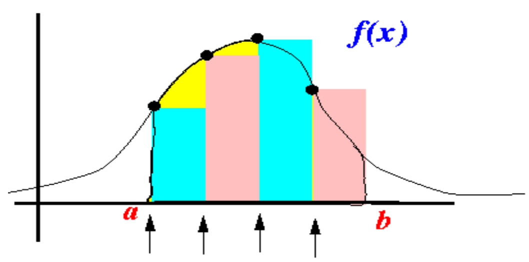



Figure 5. Rectangles as subintervals

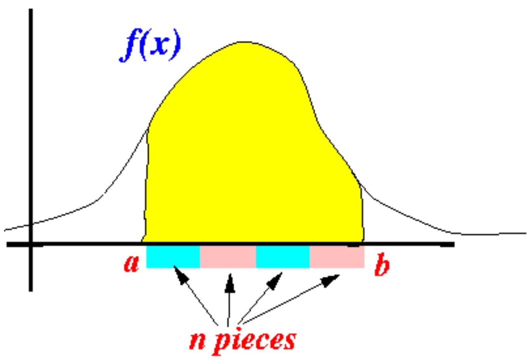



Figure. 6. Dividing in subintervals

For computing the entire interval of studying, we should summarize all the subintervals we have got.
As n gets larger, this approximation gets more accurate.
In fact, this computation is the spirit of the definition of the Riemann integral and the limit of this approximation as $n \to \infty$ is defined and equal to the integral of f on $[a,b]$ if this Riemann integral is defined.
Note that this is true regardless of which $i'$ is used, however the midpoint approximation tends to be more accurate for finite n.

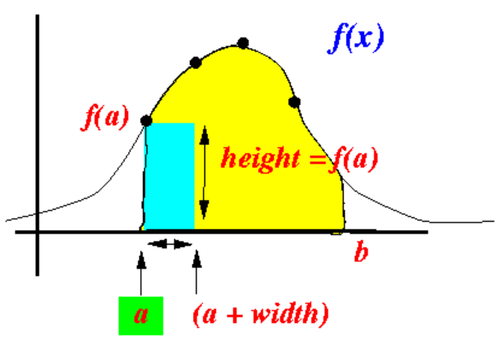



Figure 7. Computing the integral function

Numeric quadrature

We are going to take $[a,b] \subset \mathbb{R}$, x1, x2, ..., xn distinct points of [a,b] interval (a division) and $f:[a,b] \to R$, a continuous function.

By a quadrature formula we mean an quality of the type :

$$S(f) = \sum_{i=1}^{n} c_i f(x_i)$$

, where $c_i \in R$, $\forall i = \overline{1,n}$.

By numerical integration we mean an approximation of the type:

$$\int_a^b f(x)dx \cong S(f)$$

For $n \in \mathbb{N}^*$, $h = (b-a)/n$, $x_i = a + ih$, $\forall i \in \overline{0,n}$. If $S_i(f)$ is a quadrature formula of the function $f$ on the $[x_i, x_{i+1}]$ interval, $\forall i \in \overline{0,n-1}$, then it can be defined the summed quadrature formula:

$$S^{(n)}(f) = \sum_{i=0}^{n-1} S_i(f)$$

.

The rectangular quadrature formula:
The rectangular quadrature formula is:

$$S(f) = \int_a^b P\left(f, \frac{a+b}{2}, x\right)dx$$

.

Considering Newton's formula of representing polynomial interpolation, we e get:

$$P\left(f, \frac{a+b}{2}, x\right) = f\left(\frac{a+b}{2}\right)$$

Then,

$$S(f) = \int_a^b f\left(\frac{a+b}{2}\right)dx = (b-a) f\left(\frac{a+b}{2}\right)$$

If $f \in C^1([a,b])$, from the formula of evaluation of the error, at the interpolation, we will have:

$$\left| f(x) - P\left(f, \frac{a+b}{2}, x\right) \right| \le \left| x - \frac{a+b}{2} \right| \cdot \max_{x \in [a,b]} \left| f'(x) \right|$$

Then,

$$\left| \int_a^b f(x)dx - S(f) \right| = \left| \int_a^b \left( f(x) - P\left(f, \frac{a+b}{2}, x\right) \right) dx \right| \le$$

$$\int_a^b \left| f(x) - P\left(f, \frac{a+b}{2}, x\right) \right| dx \le$$

$$\le \max_{x\in[a,b]}\left|f^{'}(x)\right|\cdot\int_a^b\left|x-\frac{a+b}{2}\right|dx=\frac{(b-a)^2}{4}\cdot\max_{x\in[a,b]}\left|f^{'}(x)\right|$$
.

Observation: If $f$ is derivable in the point $\left(\frac{a+b}{2}\right)$, then the rectangular quadrature formula can be obtained by the formula:

$$S(f)=\int_a^b P\left(f,\frac{a+b}{2},\frac{a+b}{2},x\right)dx$$
.

In that case, if $f\in C^1([a,b])$, from the formula of evaluation of the error at the interpolation, we are going to have:

$$\left|f(x)-P\left(f,\frac{a+b}{2},\frac{a+b}{2},x\right)\right|\le$$

$$\le\frac{1}{2}\left|x-\frac{a+b}{2}\right|^2\cdot\max_{x\in[a,b]}\left|f^{'}(x)\right|,\ \forall x\in[a,b].$$

Then:

$$\left|\int_a^b f(x)dx-S(f)\right|=\left|\left(\int_a^b f(x)-P\left(f,\frac{a+b}{2},\frac{a+b}{2},x\right)\right)dx\right|\le$$

$$\int_a^b\left|f(x)-P\left(f,\frac{a+b}{2},\frac{a+b}{2},x\right)\right|dx\le$$

$$\le\max_{x\in[a,b]}\left|f''(x)\right|\cdot\int_a^b\left(x-\frac{a+b}{2}\right)^2dx=\frac{(b-a)^3}{24}\cdot\max_{x\in[a,b]}\left|f''(x)\right|$$

The summed quadrature formula of the rectangle is:

$$S_D^{(n)}(f)=\sum_{i=0}^{n-1}(x_{i+1}-x_i)f\left(\frac{x_i+x_{i+1}}{2}\right)=$$

$$=\frac{b-a}{n}\sum_{i=0}^{n-1}f\left(\frac{x_i+x_{i+1}}{2}\right)$$

$$=h\sum_{i=0}^{n-1}f\left(\frac{x_i+x_{i+1}}{2}\right)$$

And if $f\in C^2([a,b])$ then we will have the error evaluation :

$$\left|\int_a^b f(x)dx-S_D^{(n)}(f)\right|\le\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}f(x)dx-S_i(f)\right|\le$$

$$\le\frac{(b-a)^3}{24n}\cdot\max_{x\in[a,b]}\left|f''(x)\right|,$$

Where $S_i(f)$ is rectangular quadrature formula applied to the $f$ function on the $[x_i,x_{i+1}]$ interval $\forall i=\overline{0,n-1}$.

The trapezoidal method

In numerical analysis, the trapezoidal rule (also known as the trapezoid rule or trapezium rule) is a technique for approximating the definite integral.

$$\int_a^b f(x)dx$$

The trapezoidal rule works by approximating the region under the graph of the function $f(x)$ as a trapezoid and calculating its area. It follows that

$$\int_a^b f(x)dx\approx(b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

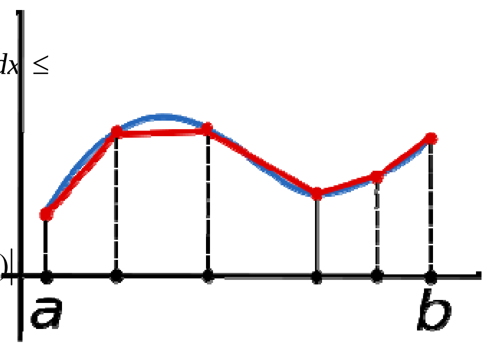



Figure 8. The trapezoidal rule

Applicability and alternatives

The trapezoidal rule is one of a family of formulas for numerical integration called Newton–Cotes formulas, of which the midpoint rule is similar to the trapezoid rule.

Moreover, the trapezoidal rule tends to become extremely accurate when periodic functions are integrated over their periods, which can be analyzed in various ways.

The quadrature trapezoidal formula is:

$$S(f)=\int_a^b P(f,a,b,x)dx$$

From Newton's formula of representing of polynomial interpolation we have the equality

$$P(a,b,f,x)=f(a)+f(a,b)(x-a)=$$

$$=f(a)+\frac{f(b)-f(a)}{b-a}(x-a).$$

Then,

$$S(f)=\int_a^b\left(f(a)+\frac{f(b)-f(a)}{b-a}(x-a)\right)dx=$$
$$=(b-a)f(a)+\frac{f(b)-f(a)}{b-a}\cdot\frac{(b-a)^2}{2}$$
$$=\frac{b-a}{2}\cdot\left(f(a)+f(b)\right).$$

If $f\in C^2([a,b])$, from error evaluation formula at the interpolation we have: $\left|f(x)-P(f,a,b,x)\right|\le$

$$\le\frac{1}{2}\left|(x-a)(x-b)\right|\cdot\max_{x\in[a,b]}\left|f''(x)\right|,\ \forall x\in[a,b].$$

Thus,

$$\left|\int_a^b f(x)dx-S(f)\right|=$$
$$=\left|\int_a^b\left(f(x)-P(f,a,b,x)\right)dx\right|\le$$
$$\le\int_a^b\left|f(x)-P(f,a,b,x)dx\right|\le$$
$$\le\frac{1}{2}\max_{x\in[a,b]}\left|f''(x)\right|\cdot\int_a^b(x-a)(b-x)dx=$$
$$=\frac{1}{2}\max_{x\in[a,b]}\left|f''(x)\right|\cdot\frac{(b-a)^3}{6}=\frac{(b-a)^3}{12}\cdot\max_{x\in[a,b]}\left|f''(x)\right|$$

The summed quadrature trapezoidal formula is:

$$S_T^{(n)}(f)=\frac{1}{2}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left(f(x_i)+f(x_{i+1})\right)=$$
$$=\frac{b-a}{2n}\sum_{i=0}^{n-1}\left(f(x_i)+f(x_{i+1})\right)$$
$$=h\sum_{i=0}^{n-1}\left(\frac{f(x_i)+f(x_{i+1})}{2}\right)$$

For which, if $f\in C^2([a,b])$, we will have the error evaluation:

$$\left|\int_a^b f(x)dx-S_T^{(n)}(f)\right|\le\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}f(x)dx-S_i(f)\right|\le$$
$$\le\frac{(b-a)^3}{12n^2}\cdot\max_{x\in[a,b]}\left|f'(x)\right|$$

where $S_i(f)$ is the trapezoidal quadrature formula of the $f$ function applied on the $[x_i,x_{i+1}]$ interval, $\forall i\in\overline{0,n-1}$

**RESULTS AND DISCUSSIONS**

1) Approximate the integral $\int_0^1 x^3dx$.

It is a very simple integral, whose primitive is calculable:

$$\int_0^1 x^3dx=\left.\frac{x^4}{4}\right|_0^1=\frac{1}{4}=0,25$$

We will also approximate it with both methods. The calculations are done with QUATTRO PRO software. If we denote $f(x)=x^3$, then $f'(x)=3x^2$ and $f''(x)=6x$, $x\in[0,1]$

Approximation 1) By rectangles method

We denote $M=\max_{x\in[0,1]}\left|f'(x)\right|=3$ and let $\varepsilon$ be the error of approximation. In order to that $\frac{(b-a)^2}{4n}\cdot M\le\varepsilon$ it suffices that

$$n\ge\left\lfloor\frac{(b-a)^2\cdot M}{4\varepsilon}\right\rfloor+1$$

In our case, for $\varepsilon=10^{-2}$ we must take $n\ge 76$.

| | |
|---|---|
| a= | 0 |
| b= | 1 |
| h= | 0.013157 |
| n= | 76 |

| Xi | (Xi+Xi+1)/2 | f((Xi+Xi+1)/2) |
|---|---|---|
| 0 | 0.006578947 | 2.847536083E-07 |
| 0.013157 | 0.006578947 | 2.847536083E-07 |
| ..... | ..... | ..... |
| 0.98684210 | 0.993421052 | 0.9803927207865 |
| 1 | | |

We obtained

$S_D^{(76)} =$ 0.249978

Approximation 2) By trapaezoids method

We denote $M = \max_{x\in[0,1]} \left| f''(x) \right| = 6$ and let $\varepsilon$ be the error of approximation. In order that $\frac{(b-a)^3}{12n^2} \cdot M \le \varepsilon$ it suffices that

$$n \ge \left\lfloor \sqrt{\frac{(b-a)^3 \cdot M}{12\varepsilon}} \right\rfloor + 1$$

In our case, for $\varepsilon = 10^{-2}$ we must take $n \ge 8$.

| | |
|---|---|
| a= | 0 |
| b= | 1 |
| h= | 0.125 |
| n= | 8 |

| Xi | f(Xi) | [f(Xi)+f(Xi+1)]/2 |
|---|---|---|
| 0 | 0 | 0.0009765625 |
| 0.125 | 0.001953125 | 0.0087890625 |
| 0.25 | 0.015625 | 0.0341796875 |
| 0.375 | 0.052734375 | 0.0888671875 |
| 0.5 | 0.125 | 0.1845703125 |
| 0.625 | 0.244140625 | 0.3330078125 |
| 0.75 | 0.421875 | 0.5458984375 |
| 0.875 | 0.669921875 | 0.8349609375 |
| 1 | 1 | |

We obtain

$S_T^{(8)} =$ 0.25390625

Remark. The trapezoids method converges much faster the the rectangles method so we will do the rest of our applications only with the trapezoids method.

2) Approximate the integral $\int_1^2 \frac{\ln(1+x)}{x} dx$.

In this case, the primitive is incalculable (transcendental), so we can't compute the integral by elementary methods. We approximate it using the trapezoids method.

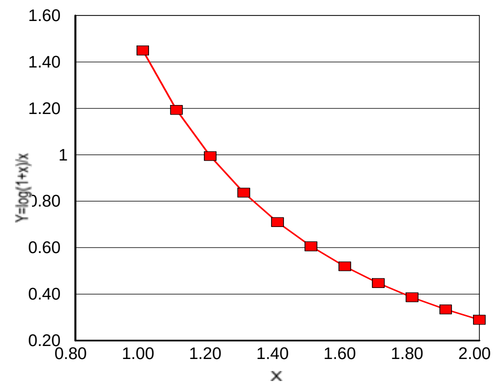
Let $f:[1,2] \to \mathbb{R}$, $f(x) = \frac{\ln(1+x)}{x}$.

We have

$$f''(x) = -\frac{1}{x^2} + \frac{1}{(x+1)^2} - \frac{1}{x(x+1)} + \frac{\ln(1+x)}{x}$$

In order to estimate the maximum value of $\left| f''(x) \right|$ we trace the graph using the CHART command in QUATTRO

| x | \|f''(x)\| |
|---|---|
| 1 | 1.45 |
| 1.1 | 1.19 |
| 1.2 | 0.99 |
| 1.3 | 0.84 |
| 1.4 | 0.71 |
| 1.5 | 0.61 |
| 1.6 | 0.52 |
| 1.7 | 0.45 |
| 1.8 | 0.39 |
| 1.9 | 0.33 |
| 2 | 0.29 |




Using the OPTIMIZER command in QUATTRO, we find.

Thus, for $\varepsilon = 10^{-5}$ we must take $n \ge 110$.

We obtain

$\int_1^2 \frac{\ln(1+x)}{x} dx \approx S_T^{(110)} =$ 0.26678

**3)** *Approximate the integral* $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sin x}{x} dx$.

The primitive is transcendental.
Let

$$f:\left[\frac{\pi}{3},\frac{\pi}{2}\right]\to \mathbb{R},\ f(x)=\frac{\sin x}{x}.$$

We have

$$f''(x)=\frac{-x^3\sin x-2x^2\cos x+2x\sin x}{x^4}$$

We find

$$M=\max_{x\in[0,1]}\left|f''(x)\right|=0.23.$$

Thus, for $\varepsilon=10^{-5}$ we must take $n\geq 15$.
We obtain

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{\sin x}{x}dx \approx S_T^{(15)}=0.38529$$

**4)** *Approximate the integral* $\int_0^1 e^{-x^2}dx$.

The primitive is transcendental.
Let

$$f:[0,1]\to \mathbb{R},\ f(x)=e^{-x^2}.$$

We have

$$f''(x)=\left(-2+4x^2\right)e^{-x^2}$$

We find

$$M=\max_{x\in[0,1]}\left|f''(x)\right|=2.$$

Thus, for $\varepsilon=10^{-5}$ we must take $n\geq 130$.
We obtain

$$\int_0^1 e^{-x^2}dx \approx S_T^{(130)}=0.74682$$

*Remark.*(see http://en.wikipedia.org/wiki/Normal_distribution)

In statistics $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2}}$ is the density function of the normal distribution $N(\mu,\sigma^2)$ (with the mean $\mu$ and the standard deviation $\sigma$). Many random variables or fenomenons have a normal distribution, whose graph is also known as the bell of Gauss: the marks from a test, people's hights, people's IQ etc.

The probability ho have values in $[a,b]$ is

$$P\left(a\leq N(\mu,\sigma^2)\leq b\right)=\int_a^b\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2}}dx$$

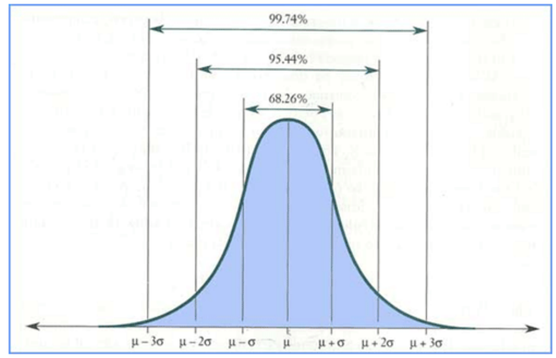



Figure 9. The Gaussian bell

An integral which can be approximated in the same manner as before.

**5)** *Approximate the integral* $\int_0^{2\pi}\sqrt{1+3\sin^2 x}dx$.

(an elliptic integral)

The primitive is transcendental.
Let

$$f:[0,2\pi]\to \mathbb{R},\ f(x)=\sqrt{1+k\sin^2 x},\ k\geq 0.$$

We have

$$f''(x)=\frac{4k^2\cos^2 2x+(2k^2+4k)\cos 2x-2k^2}{4\left(\frac{2+k-k\cos 2x}{2}\right)^{\frac{3}{2}}}$$

We could have used QUATTRO for computing the maximum, but, using modul's properties, we found the general inequality

$$\left|f''(x)\right|\leq 2k^2+k,\ x\in[0,2\pi]$$

So, for k=3

$$M=\max_{x\in[0,2\pi]}\left|f''(x)\right|<21$$

For $\varepsilon=10^{-5}$ we should have taken $n\geq 3000$, so, for the sake of simplicity, we took $\varepsilon=10^{-2}$, for which it suffices to consider $n\geq 209$.
We obtain

$$\int_0^{2\pi}\sqrt{1+3\sin^2 x}dx \approx S_T^{(209)}=9.68844$$

*Remark.* (see http://en.wikipedia.org/wiki/Ellipse).

Using, for example, line integrals the circumference of an ellipse with the semi-axes $a > b > 0$ is

$$\oint_E ds = b\int_0^{2\pi} \sqrt{1+\left(\frac{a^2}{b^2}-1\right)\sin^2 x dx}$$

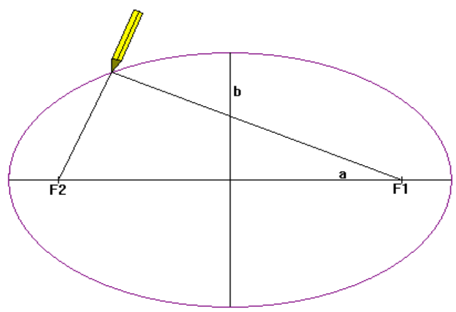



Figure 10. An ellipse

Unfortunately, this integral, called *elliptic*, is transcendental, so we can only approximate it.

Taking $\frac{a^2}{b^2}-1=k\geq 0$, we have to evaluate

$$\int_0^{2\pi} \sqrt{1+k\sin^2 x dx}$$

In fact, we see that our last calculation represents an aproximation for the circumference of the ellipse with $a=2$ and $b=1$.

Our result is very close to Ramanujan's approximation:

$$C \approx \pi\left[3(a+b)-\sqrt{(3a+b)(a+3b)}\right]=9.6884210$$

## CONCLUSIONS

We succesfully managed to calculate (approximatively) some important integrals, mainly with transcendental primitives. Rectangles formula is a a little easier to apply, esspecialy if we take the the extremities instead of the middle, but trapezoids formula is faster. Although, there exist even faster methods. Sometimes, in order to get a good approximation, we must take a big valule for n, which makes the problem difficult in QUATTRO or EXCEL. In this cases, a programming language or routine would be better.

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