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SRINIVASA RAMANUJAN – THE MAN WHO (WAS?) REVEALED INFINITY

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Srinivasa Ramanujan (1887-1920)

Abstract

The Indian mathematician Srinivasa Ramanujan (22 December 1887 – 26 April 1920) was one of the most enigmatic figures in the history of science. He was born in a poor family and had no formal education in mathematics, being mostly self-taught, but he made important contributions to mathematical analysis, number theory, infinite series, and continued fractions, including the solutions to some open mathematical problems. Some of his work even has applications in nowadays top science domains, such as cosmology and string theory. Back then, paper was very expensive and Ramanujan did most of his calculations with chalk on the ground floor of a local temple and then recorded them in his notebooks, mostly without the proofs, just like in the book "A Synopsis of Elementary Results in Pure and Applied Mathematics" by G. S. Carr, which influenced him deeply. He said that his formulas were revealed to him in his dreams by some Indian gods, especially by the goddess Namagiri. In his short life he found more than 3900 results (mostly identities and equations), many of them completely novel and very unconventional, which opened new directions in science and were inspiring for later scientists.

***Key words**: continued fractions, infinite series, mathematical analysis, number theory, Srinivasa Ramanujan.*

INTRODUCTION

EARLY LIFE

In 2020 the scientific world commemorated the centenary of the great Indian mathematician Srinivasa Ramanujan’s passing away. Ramanujan was born on 22 December 1887 in Erode, Tamil Nadu, at the residence of his maternal grandparents. They lived in Sarangapani Street in a traditional home in the town of Kumbakonam. In December 1889, Ramanujan had smallpox and recovered. He moved with his mother to her parents’ house in Kanchipuram, near Chennai. Since Ramanujan's father was at work most of the day, his mother took care of him as a child. From her, he learned about tradition and puranas (Hindu religious texts that are part of the Vedas). He has been taught to be a religious person and he was praying every day. Just before the age of 10, in November 1897, he passed his primary examinations in English, Tamil, geography and arithmetic. With his scores, he stood first in the district. That year, Ramanujan entered Town Higher Secondary School where he encountered formal mathematics for the first time. By the age of 11, he had exhausted the mathematical knowledge of two college students who were lodgers at his home. He was later lent a book on advanced trigonometry written by S. L. Loney. He completely mastered this book by the age of 13 and discovered sophisticated theorems on his own. By 14, he was receiving merit certificates and academic awards which continued throughout his school career and also assisted the school in the logistics of assigning its 1200 students (each with their own needs) to its 35-odd teachers. He completed mathematical exams in half the allotted time, and showed a familiarity with geometry and infinite series. Ramanujan was shown how to solve cubic equations in 1902 and he went on to find his own method. In 1903 when he was 16, Ramanujan obtained from a friend a library-loaned copy of the book “A Synopsis of Elementary Results in Pure and Applied Mathematics” by G. S. Carr, which contained a collection of 5000 theorems. Ramanujan reportedly studied the contents of the book in detail. The book is generally acknowledged as a key element in awakening the genius of Ramanujan. The next year, he had independently developed and studied the Bernoulli numbers and had calculated Euler's constant up to 15 decimals. When he graduated from Town Higher Secondary School in 1904, Ramanujan was awarded the “K. Ranganatha Rao” prize for mathematics by the school's headmaster, Krishnaswami Iyer, who introduced Ramanujan as an outstanding student who deserved scores higher than the maximum possible marks. He received a scholarship to study at Government Arts College, Kumbakonam, However, Ramanujan was so intent on studying mathematics that he could not focus on any other subjects and failed most of them, losing his scholarship in the process. He later enrolled at Pachaiyappa's College in Madras. He again excelled in mathematics but performed poorly in other subjects such as physiology. Ramanujan failed his Fine Arts degree exam in December 1906 and again a year later. Without a degree, he left college and continued to pursue independent research in mathematics. At this point in his life, he lived in extreme poverty and was suffering from starvation.

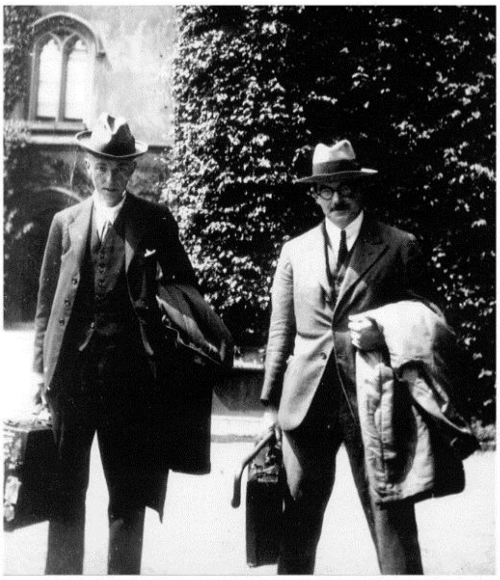
ADULTHOOD IN INDIA

On 14 July 1909, Ramanujan was married to a nineyear old bride, Janaki Ammal. After the marriage, Ramanujan developed a hydrocele testis. His family did not have the money for the operation, but in January 1910, a doctor volunteered to do the surgery for free. After his successful surgery, Ramanujan searched for a job. He stayed at friends' houses while he went door to door around the city of Chennai looking for a clerical position. To make some money, he tutored some students at Presidency College who were preparing for their exam.

MATHEMATICAL ACHIEVEMENTS

In mathematics, there is a distinction between having an insight and having a proof. Ramanujan's talent suggested a group of formulae that could then be investigated in depth later. It is said that Ramanujan's discoveries are unusually rich. As a by-product, new directions of research were opened up. Examples of the most interesting of these formulae include the infinite series for π, one of which is given below One of his remarkable capabilities was the rapid solution for problems. He was sharing a room with P. C. Mahalanobis who had a problem, "Imagine that you are on a street with houses marked 1 through n. There is a house in between (x) such that the sum of the house numbers to left of it equals the sum of the house numbers to its right. If n is between 50 and 500, what are n and x?" This is a bivariate problem with multiple solutions. Ramanujan thought about it and gave the answer with a twist: He gave a continued fraction. The unusual part was that it was the solution to the whole class of problems. Mahalanobis was astounded and asked how he did it. "It is simple. The minute I heard the problem, I knew that the answer was a continued fraction. Which continued fraction, I asked myself. Then the answer came to my mind," Ramanujan replied. In 1918, Hardy and Ramanujan studied the partition function $p(n)$ extensively and gave a nonconvergent asymptotic series that permits exact computation of the number of partitions of an integer. He discovered mock theta functions in the last year of his life. For many years these functions were a mystery, but they are now known to be the holomorphic parts of harmonic weak Maass forms.

RAMANUJAN, HARDY AND LITTLEWOOD



G.H. Hardy (left) and J.E. Littlewood (right)

Since 1904, Ramanujan started to produce sophisticated mathematical results. After his introduction in 1910 to V. Ramaswamy Aiyer, the founder of the Indian Mathematical Society, he began to publish his results in the Society's Journal.

In 1913, Ramanujan wrote a letter to three proeminent English mathematicians from Cambridge University. The first two, H. F. Baker and E. W. Hobson returned the paper without any comment. On 2 February 1913, England's leading mathematician Professor G.H. Hardy, from Trinity College, Cambridge University, opened an untidy envelope with many Indian postage stamps to find on crumpled pages an unsolicited letter, dated 16 January 1913, from an unknown Hindu clerk, which began:

"Dear Sir, I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £20 per annum. I am now about 23 years of age. I have had no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course followed in an University course, but I am striking out a new path for myself. I have made special investigations of divergent series in general, and the results I get are termed by the local mathematicians as "startling".

There followed eleven pages of mathematical formula, wild and fantastic theorems on prime numbers, infinite series, integrals, and continued fractions (for those who seek a copy of these pages, note that pages 8 and 10 are lost). Hardy's first reaction was one of irritation at the large number of theorems stated without proof: perhaps the author was a crank, or the letter an elaborate fraud; or perhaps this was a well-crafted practical joke by a colleague. However, Hardy was intrigued, and at the end of his typical and routine Cambridge day his second reaction was to meet with his colleague J.E. Littlewood and to work as best they could through the theorems. Some were already theorems such as the integral formula (Berndt, 1997).

$$\int_0^a e^{-x^2}dx = \frac{\sqrt{\pi}}{2} - \cfrac{e^{-a^2}}{2a + \cfrac{1}{a + \cfrac{2}{2a + \cfrac{3}{a + \cfrac{4}{2a + \cdots}}}}}$$

Others were new, but rather interesting than important results, such as

$$\int_0^\infty \frac{\cos \pi x}{\Gamma^2(\alpha + x)\Gamma^2(\alpha - x)}dx = \frac{1}{4\Gamma(2\alpha - 1)\Gamma^2(\alpha)}$$
$$\text{where} \quad \alpha > \frac{1}{2}$$

And yet others were nothing like anything that they had seen before. For example

$$\cfrac{1}{1 + \cfrac{e^{-2\pi}}{1 + \cfrac{e^{-4\pi}}{1 + \cdots}}} = \left(\sqrt{\frac{5 + \sqrt{5}}{2}} - \frac{\sqrt{5} + 1}{2}\right) e^{\frac{2}{5}\pi}$$

Hardy said that some of the formulas had "defeated" him "completely".

"I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them".

Hardy and Littlewood showed Ramanujan's results to some of their colleagues, and it quickly became obvious that Ramanujan was a mathematician of exceptional ability. They decided to invite him to Britain in order to receive formal training and an advanced degree from Cambridge. However, Ramanujan was initially unwilling to leave his family or to cross the sea for religious reasons; so, as a short-term measure, his mathematical friends in India applied, successfully, for a temporary research scholarship on his behalf at the University of Madras, which allowed him to concentrate on his mathematical research full time while supporting his family financially. But there was still the matter of persuading him to make the long sea journey to England. After a year of wrangling, these obstacles were gradually surpassed and, with his family's consent, he finally agreed to go, setting sail for England on 17 March 1914.

On his arrival in Cambridge the following month, he immediately began to work with Hardy and Littlewood. Not surprisingly, however, they soon found that their mathematical methodologies were profoundly different. Hardy and Littlewood, being strict analysts, were insistent on absolute rigour and formal proofs, while Ramanujan was content to rely on intuition and inductive experimentation. Indeed, as Littlewood later wrote

"He was not interested in rigour … [and] the clear-cut idea of what is meant by a proof … he perhaps did not possess at all."

Also, talking about Ramanujan's intuition, he noted that "every positive integer was one of [Ramanujan's] personal friends."

Hardy thus decided to remedy the gaps in Ramanujan's formal training by sending him to take some courses, without destroying his enthusiasm and wild intuition. This proved to be a real challenge.

Adapting to life in Britain was difficult for Ramanujan. The cold and wet climate affected him, and he had problems getting vegetarian food, especially after the beginning of the First World War in August 1914. He cooked his own simple meals in his room in Trinity College, and thus did not eat with the rest of the college community as would usually have been expected. The war also resulted in many of Cambridge's best mathematicians, including Littlewood, being called away for military service, further adding to Ramanujan's sense of isolation.

Despite all the problems, Ramanujan's mathematical output was rapid. He quickly began to publish papers in British journals, including a 60-page paper in the Proceedings of the London Mathematical Society in 1915 on "highly composite numbers, for which he was awarded the degree of Bachelor of Science by Research (renamed PhD in 1920) by the University of Cambridge.

During his stay in England, Ramanujan published alone or in collaboration 21 articles.

He was elected member of the London Mathematical Society in December 1917, followed by fellowships of the Royal Society in May 1918 and Trinity College in October 1918.

Ramanujan’s health had never been strong, but by May 1917 he was seriously ill, being diagnosed with tuberculosis and severe vitamin deficiency, although recent investigations have suggested that he may have been suffering from hepatic amoebiasis, a complication arising from previous attacks of dysentery. By November 1918, his health had improved enough for Hardy to write about a return to his home land:

“He will return to India with a scientific standing and reputation such as no Indian has enjoyed before, and I am confident that India will regard him as the treasure he is. His natural simplicity and modesty have never been affected in the least by success – indeed all that is wanted is to get him to realise that he really is a success”.

On 27 February 1919, he embarked for India, arriving in Kumbakonam two weeks later, but his health deteriorated again despite medical treatment. He died on 26 April 1920 at the age of 32.

RAMANUJAN'S NOTEBOOKS

Ramanujan recorded the bulk of his results in four notebooks of loose - leaf paper. These results were mostly written up without any proofs. This is probably the origin of the misperception that Ramanujan was unable to prove his results and simply thought up the final result directly. This style of working may have been for several reasons. Since paper was very expensive, Ramanujan would do most of his work and perhaps his proofs on slate, and then transfer just the results to paper. Using a slate was common for mathematics students in the Madras Presidency at the time. His first notebook has 351 pages with 16 somewhat organized chapters and some unorganized material. The second notebook has 256 pages in 21 chapters and 100 unorganised pages, with the third notebook containing 33 unorganised pages. The results in his notebooks inspired numerous papers by later mathematicians trying to prove what he had found. Hardy himself created papers exploring material from Ramanujan's work as did G. N. Watson, A fourth notebook with 87 unorganised pages, the so called "lost notebook", was rediscovered in 1976 by George Andrews. Notebooks 1, 2 and 3 were published as a two volume set in 1957 by the Tata Institute of Fundamental Research (TIFR), Mumbai, India. This was a photocopy edition of the original manuscripts, in his own handwriting. In December 2011, as part of Ramanujan's 125th birth centenary celebrations, TIFR republished the notebooks in a coloured two volume collector's edition. These were produced from scanned and microfilmed images of the original manuscripts by expert archivists of Roja Muthiah Research Library, Chennai.

RECOGNITION

Ramanujan's home state of Tamil Nadu celebrates 22 December (Ramanujan's birthday) as 'State IT Day', memorializing both the man and his achievements, as a native of Tamil Nadu. A stamp picturing Ramanujan was released by the Government of India in 1962 – the 75th anniversary of Ramanujan's birth – commemorating his achievements in the field of number theory and a new design was issued on December 26, 2011, by the India Post. Since the Centennial year of Ramanujan, every year 22 Dec, is celebrated as Ramanujan Day by the Government Arts College, Kumbakonam where he had studied and later dropped out. Ramanujan's work and life are celebrated on 22 December at The Indian Institute of Technology (IIT), Madras in Chennai. A prize for young mathematicians from developing countries has been created in the name of Ramanujan by the International Centre for Theoretical Physics (ICTP), in cooperation with the International Mathematical Union, who nominate members of the prize committee. The Shanmugha Arts, Science, Technology & Research Academy (SASTRA), based in the state of Tamil Nadu in South India, has instituted the SASTRA Ramanujan Prize of $10,000 to be given annually to a mathematician not exceeding the age of 32 for outstanding contributions in an area of mathematics influenced by Ramanujan. The age limit refers to the years Ramanujan lived, having nevertheless still achieved many accomplishments. This prize has been awarded annually since 2005, at an international conference conducted by SASTRA in Kumbakonam, Ramanujan's hometown, around Ramanujan's birthday, 22 December. On the 125th anniversary of his birth, India declared the birthday of Ramanujan, December 22, as 'National Mathematics Day' (Karthikeyan, 2018).

RAMANUJAN’S MATHEMATICAL LEGACY

During his short life, Ramanujan independently obtained nearly 3,900 results (mostly identities and equations). Many were completely novel; his original and highly unconventional results, such as the Ramanujan prime, the Ramanujan theta function, partition formulae and mock theta functions, have opened entire new areas of work and inspired a vast amount of further research. Nearly all his claims have now been proven correct (Berndt, 1997; Merrotsy, 2020).

Sum of consecutive integers

$$1+2+3+\cdots=-\frac{1}{12}$$

This claim is obviously false if one considers the classical convergence of sequences and series in real analysis. It can be “proved” by considering that this serie and others similar converge in a different way. However, the result has a deep meaning. The most important function in number theory is the *Riemann Zeta function*

$$\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}$$

Note that, formally,

$$\zeta(-1)=\sum_{n\geq 1}\frac{1}{n^{-1}}=1+2+3+\cdots$$

$\zeta(s)$ converges for $s>1$, but it can be extended analiticaly to all complex numbers, and the value of its extension at $s=-1$ is indeed $-\frac{1}{12}$.

Partition formula (Hardy-Ramanujan-Rademacher asymptotic formula)

This formula helped in establishing the fame of Ramanujan in the world of mathematics. He along with Hardy discovered and proved an asymptotic formula for the well known partition function which helps to approximate the function for large enough numbers. This function also has a lot of applications both within number theory and outside.

The *partition number* $p(n)$ represents the number of ways in which a positive integer n can be written as a sum of positive descending integers. So, for instance, the integer $n=5$ can be written in five different ways, namely: 5, $4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$. Thus, $p(5)=7$. Other examples*:* $p(15)=176$, $p(34)=12310$ and

$$p(200)=3\,972\,999\,029\,388\,004.$$

In 1918 Hardy and Ramanujan proved the asympthotic formula

$$p(n)\sim\frac{1}{4n\sqrt{3}}\,e^{\pi\sqrt{\frac{2n}{3}}}.$$

A complete asymptotic expansion was given in 1937 by Rademacher.

Ramanujan congruences

Ramanujan independently discovered three congruences related to the partition function. These are perhaps the most beautiful results given by him. In his paper (Ramanujan, 1921), the first two congruences were proved and the third was extracted from his notebooks and proved by Hardy after Ramanujan’s death.

$$p(5n+4)\equiv 0\ (mod\ 5)$$
$$p(7n+5)\equiv 0\ (mod\ 7)$$
$$p(11n+6)\equiv 0\ (mod\ 11)$$

Ramanujan master theorem

This theorem is one of the most applied results of Ramanujan. This is used to evaluate the Mellin transforms of several complex valued functions. This has been generalised and now this is used in Quantum mechanics through Feynmann diagrams.

The result states that if a function $f:\mathbb{R}\to\mathbb{C}$ admits an expansion of the form

$$f(x)=\sum_{n=0}^{\infty}\frac{\varphi(n)}{n!}\,(-x)^n$$

then the *Mellin transform* of $f(x)$ is given by

$$\int_0^\infty x^{s-1}f(x)dx=\Gamma(s)\,\varphi(-s)$$

where $\Gamma(s)=\int_0^\infty x^{s-1}e^{-x}dx\,,\ \ s\in\mathbb{C}$, $Re(s)>0$, is the *Gamma function*.

Theta functions and their results

Ramanujan greatly studied a group of functions which he named as the theta functions. These functions have been found to have a large number of properties which are applicable in advanced physics and mathematics. For example, these functions are used in the working of nuclear reactors. The Ramanujan theta function is used to determine the critical dimensions in Bosonic string theory, superstring theory and M - theory.

The *Ramanujan theta function* is defined in (Ramanujan, 1916) as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}}\, b^{\frac{n(n-1)}{2}}, \qquad for\ |ab| < 1$$

The *Jacobi triple product*

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2m+1}y^2)\left(1+\frac{x^{2m-1}}{y^2}\right) = \sum_{n=-\infty}^{\infty} x^{n^2}\, y^{2n}$$

can be written in the following way

$$f(a,b) = (-a,ab)_\infty\,(-b,ab)_\infty\,(ab,ab)_\infty$$

where $(a,q)_\infty$ denotes the q - *Pochhammer symbol*

$$(a,q)_\infty = \prod_{k=0}^{n-1}(1-aq^k)$$

Mock theta functions

Another class of functions discovered and studied by him. These are even more advanced and applicable than theta functions. Ramanujan described them in his last letter to Hardy. These functions are extremely useful in Quantum field theory and string theory. These functions are used for predicting the entropy of black holes.

Ramanujan's approximation for π

The most efficient approximation for π till date. Modern computers use this formula to approximate π up to trillions of decimal places. In 1910, Ramanujan found several rapidly converging series of π, including

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!\,(1103+26390n)}{(n!)^4\cdot 396^{4n}}$$

which computes 8 new decimals of π with each term.

Ramanujan's results on highly composite numbers

Ramanujan got his BSc degree after writing his paper entitled “Highly composite numbers” (HCN). A *highly composite number* (or anti-prime number) is a positive integer with more divisors than any of the numbers smaller than it. Such examples are: 1, 2, 4, 6, 12 etc. He discovered these numbers and studied them and as a result discovered highly original results of the subject.

Generalization of Bertrand's postulate

Bertrand's postulate states that for any n≥3 there exists at least a prime number p, such that

$$n < p < 2n$$

In other words, if we denote for $x \in \mathbb{R}$ the prime counting function

$$\pi(x) = card\,\{p \le x\,;\ \ p\ \ prime\}$$

then Bertrand’s postulate can be reformulated:

$$\pi(2n) > \pi(n), \qquad \forall\, n \ge 3$$

One of the three mathematicians who proved the Bertrand's postulate was Ramanujan. His short proof involved advanced properties of the gamma function and in this proof, he generalized the result which is the best generalization till date. Later on, his methods of generalization were used by several mathematicians like Sondow who first defined and studied the Ramanujan primes.

Ramanujan derives the postulate from the following stronger inequality:

$$\pi(x) - \pi\left(\frac{x}{2}\right) > \frac{1}{\log x}\left(\frac{x}{6} - 3\sqrt{x}\right), \qquad \forall\, x > 300$$

A conjecture of Ramanujan’s related to Fermat’s Last Theorem

In his paper "On certain arithmetical functions" (Ramanujan, 1916), Ramanujan defined the so-called *delta-function*, whose coefficients are called $\tau(n)$ (the *Ramanujan tau function*).

He proved many congruences for these numbers, such as:

$$\tau(p) \equiv 1 + p^{11} (mod\ 691), \qquad for\ primes\ p$$

This congruence (and others like it that Ramanujan proved) inspired Jean-Pierre Serre (1954 Fields Medalist) to conjecture that there is a theory of Galois representations that "explains" these congruences and more generally all modular forms. Δ(z) is the first example of a modular form to be studied in this way. Deligne (in his Fields Medal-winning work in 1973) proved Serre's conjecture. That Ramanujan conjecture is an assertion on the size of the tau-function, which has as generating function the discriminant modular form Δ(q), a typical cusp form in the theory of modular forms. The proof of Fermat's Last Theorem proceeds by first reinterpreting elliptic curves and modular forms in terms of these Galois representations. Without this theory there would be no proof of Fermat's Last Theorem.

Hardy – Ramanujan number 1729

"I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. "No", he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways." (Hardy)

The two ways are

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

Generalisations are now called "taxicab numbers".

CONCLUSIONS

Ramanujan's mathematical achievements and life have inspired generations of mathematicians and scientists from India and from all over the world. Despite his extremely short life, he remains one of the most productive mathematicians throughout history.

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The Man Who Knew Infinity, a 2015 film based on Kanigel's book